Title	Well-posedness of the Cauchy problem for the KdV equation at the critical regularity (Harmonic Analysis and Nonlinear Partial Differential Equations)
Author(s)	Kishimoto, Nobu
Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2010), B18: 57-73
Issue Date	2010-06
URL	http://hdl.handle.net/2433/176865
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Well-posedness of the Cauchy problem for the KdV equation at the critical regularity

京都大学大学院理学研究科　岸本 展 (Nobu Kishimoto)
Department of Mathematics, Kyoto University

1 Introduction

This note is a survey of the author's papers [19, 20] on the Cauchy problem for the *Korteweg-de Vries (KdV) equation*:

$$\begin{cases} \partial_t u + \partial_x^3 u = \partial_x(u^2)\ , \qquad u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R} \text{ or } \mathbb{C}, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

The KdV equation, which was originally derived by Korteweg and de Vries [21] as a model equation for the propagation of shallow water waves along a canal, appears in various phases of mathematical physics. It is also well-known as one of the simplest partial differential equations that have complete integrability.

Our principal aim is to prove the *local well-posedness* (LWP for short) of (1) in a particular Sobolev space $H^{-3/4}(\mathbb{R})$. Here, we mean by LWP in H^s that for any $r > 0$, there exists a time $T = T(r) > 0$ such that for any initial data u_0 with H^s-norm less than r, we can find a solution to the Cauchy problem in $C_t([0,T]; H^s_x)$, which is unique in some sense and depends continuously on the initial datum. As a corollary, we will obtain the global well-posedness (that is, the above T can be any large number; GWP for short) of (1) in the real-valued setting in $H^{-3/4}(\mathbb{R})$, and also GWP of the Cauchy problem for the *modified Korteweg-de Vries (mKdV) equation*:

$$\begin{cases} \partial_t v + \partial_x^3 v = \pm\partial_x(v^3)\ , \qquad v : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}\ , \\ v(0,x) = v_0(x) \end{cases} \tag{2}$$

in $H^{1/4}(\mathbb{R})$. The mKdV equation is linked with the KdV equation through the *Miura transform* $v \mapsto u := c_1\partial_x v + c_2 v^2$. In fact, if v is a smooth solution to the defocusing (resp. focusing) mKdV equation ('$+$' (resp. '$-$') sign before the nonlinear term), then $u := \frac{3}{\sqrt{2}}\partial_x v + \frac{3}{2}v^2$ (resp. $u := \frac{3i}{\sqrt{2}}\partial_x v - \frac{3}{2}v^2$) solves the real-valued (resp. complex-valued) KdV equation. The Miura transform behaves roughly like a derivative, so the result for mKdV at a certain regularity is similar to that for KdV at one lower regularity.

We shall establish LWP by means of the *iteration method*. For the KdV equation, we first replace (1) with the integral equation

$$u(t) = e^{-t\partial_x^3}u_0 + \int_0^t e^{-(t-t')\partial_x^3}[\partial_x(u(t')^2)]\,dt', \quad (\ =: \Phi(u)(t)\ )$$

and then show that Φ is a contraction mapping on a certain Banach space S_T included in $C_t([0,T];H_x^s)$ with some $T>0$. To do this, the following linear and bilinear estimates are basically needed:

$$\|e^{-t\partial_x^3}u_0\|_{S_T} \le C\|u_0\|_{H^s}, \quad \Big\|\int_0^t e^{-(t-t')\partial_x^3}[\partial_x(u(t')v(t'))]\,dt'\Big\|_{S_T} \le C\|u\|_{S_T}\|v\|_{S_T}.$$

We often separate the second estimates into the following:

$$\Big\|\int_0^t e^{-(t-t')\partial_x^3}F(t')\,dt'\Big\|_{S_T} \le C\|F\|_{N_T}, \quad \|\partial_x(uv)\|_{N_T} \le C\|u\|_{S_T}\|v\|_{S_T}$$

with some Banach space for nonlinearity N_T. It is important how to design the working function space S_T (and also N_T) for these estimates. In particular, the last bilinear estimate is usually crucial and the hardest to establish, especially in low regularity.

1.1 Known results

The Cauchy problem (1), (2) have been extensively studied. For the LWP results by the iteration method, we first recall that Kenig, Ponce, and Vega [14] obtained LWP in $H^s(\mathbb{R})$, with $s>3/4$ for (1) and with $s\ge 1/4$ for (2). The proof was achieved with the local smoothing and the maximal function estimates for the Airy operator $e^{-t\partial_x^3}$.

Well-posedness theory for the KdV equation has made substantial progress since Bourgain [2] introduced the Fourier restriction method, namely the iteration in the Bourgain spaces $X^{s,b}$ defined by

$$X^{s,b} := \big\{\, u\in\mathcal{S}'(\mathbb{R}^2) \,\big|\, \|u\|_{X^{s,b}} := \big\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\widehat{u}\big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)} < \infty \,\big\},$$

where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\widehat{u}$ denotes the spacetime Fourier transform of u. The previous result was improved to $s\ge 0$ in [2] with a new bilinear estimate in the

Bourgain spaces. Kenig, Ponce, and Vega continued the analysis in these spaces and established the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}$$

with some $b > 1/2$, first for $s > -5/8$ ([15]), and for $s > -3/4$ later ([16]). LWP in the corresponding regularity follows from this estimate. Thus, it is natural to try to establish the above bilinear estimate for $s \leq -3/4$ if one wishes to obtain the well-posedness for that regularity.

However, it is known that when $s < -3/4$ the data-to-solution operator fails to be uniformly continuous as a map from H^s to $C_t(H^s_x)$; see [17, 3]. They also showed the same result for the mKdV equation for $s < 1/4$. Although such results do not necessarily imply the ill-posedness of the Cauchy problem, the uniform continuity of the solution operator is a necessary condition for the validity of the iteration method.

We now focus on the remaining case $s = -3/4$. Unfortunately, it was shown in [16, 23] that the crucial bilinear estimate in the Bourgain space does not hold at this regularity for any $b \in \mathbb{R}$. Therefore, to obtain the remaining LWP in $H^{-3/4}$, we have to iterate in a space different from $X^{s,b}$, or abandon the direct iteration method. The latter approach was taken in [3]. They obtained the existence result for (1) in $H^{-3/4}$ by combining (slightly modified) Miura transform with the corresponding LWP for the mKdV equation in $H^{1/4}$ obtained in [14].

1.2 I-method

In general, the global well-posedness is obtained by pasting the local results. However, the basic LWP, which gives the existence time $T \sim \|u_0\|^{-\alpha}$ with some $\alpha > 0$ and the estimate $\sup_{0 \leq t \leq T} \|u(t)\| \leq C\|u_0\|$, is not sufficient by itself, because in each step, the initial datum may grow exponentially and provide the exponentially-decaying existence time. Therefore, we need some *a priori* estimate on the growth of the solution which bounds the data uniformly in each step. For instance, the L^2 conservation of the real-valued KdV solution together with LWP in L^2 immediately yields GWP of (1) in L^2 in the real-valued setting. Similarly, GWP of (2) in H^1 can be obtained from the conservation of Hamiltonian and the L^2-norm, and the local results in H^1.

However, we have no conservation law below these regularities; $0 > s \geq -3/4$ for KdV and $1 > s \geq 1/4$ for mKdV. In this situation, the so-called "I-method" introduced by Colliander, Keel, Staffilani, Takaoka, and Tao plays a great role in constructing global solutions. For real-valued (1), they first constructed an almost conserved quantity to obtain GWP for $s > -3/10$ ([6]), and then introduced some correction terms to improve the result to the same regularity as the above LWP, $s > -3/4$ ([7]). This result was also combined with the Miura transform to obtain GWP of (2) for $s > 1/4$. The I-method has been applied to GWP for a variety of nonlinear evolution equations and other topics; see [24] and references therein.

Now we recall the result for KdV in [6] to outline the I-method. Let $N \gg 1$ and $s < 0$. We define $\mathcal{I} = \mathcal{I}_{s,N}$ as the spatial Fourier multiplier with the symbol $m_{s,N}(\xi) = m_s(|\xi|/N)$, where $m_s(r) : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth monotone function which equals 1 for $r \le 1$ and r^s for $r \ge 2$.

We have the estimate $C^{-1}\|\varphi\|_{H^s} \le \|\mathcal{I}\varphi\|_{L^2} \le CN^{-s}\|\varphi\|_{H^s}$. Furthermore, the following variant of the LWP result holds for $s > -3/4$.

Lemma 1 ([6])**.** *Let $s > -3/4$. Then, there exists $b > 1/2$ such that for any $u_0 \in H^s$, a solution $u(t) \in C_t([0,\delta]; H^s_x)$ to (1) exists on $[0,\delta]$ with $\delta \ge c\|\mathcal{I}u_0\|_{L^2}^{-\alpha}$ and satisfies $\|\mathcal{I}u\|_{X^{0,b}_\delta} \le C\|\mathcal{I}u_0\|_{L^2}$. Here c, C, and α are some positive constants, and*

$$\|u\|_{X^{0,b}_\delta} := \inf\big\{ \|v\|_{X^{0,b}} \,\big|\, v \in X^{0,b},\ u(t) = v(t) \text{ for } 0 \le t \le \delta \big\}.$$

Another important feature of the operator $\mathcal{I}$ is the *almost conservation* of $\|\mathcal{I}u(t)\|_{L^2}$.

Lemma 2 ([6])**.** *Let $u(t)$ be a real-valued solution to the KdV equation on the time interval $[0,\delta]$. Then, for any $\varepsilon > 0$ and $b > 1/2$ there exists $C > 0$ independent of N such that*

$$\|\mathcal{I}u(t)\|_{L^2}^2 \le \|\mathcal{I}u(0)\|_{L^2}^2 + CN^{-\frac{3}{4}+\varepsilon}\|\mathcal{I}u\|_{X^{0,b}_\delta}^3$$

for $0 \le t \le \delta$.

We see that the quantity $\|\mathcal{I}u(t)\|_{L^2}$ is not bounded, but it may grows polynomially, rather than exponentially. In fact, it follows from Lemmas 1 and 2 that if $s > -3/4$ and the real-valued initial datum u_0 satisfies $\|\mathcal{I}u_0\|_{L^2} \le 1$, then we can iterate the local theory $O(N^{\frac{3}{4}-\varepsilon})$ times until the norm $\|\mathcal{I}u(t)\|_{L^2}$ becomes greater than 2. We thus obtain solutions at least on $[0, O(N^{\frac{3}{4}-\varepsilon})]$ from such initial data.

For general data, we utilize the scaling argument. The KdV equation is invariant under the scaling transform

$$u(t,x) \quad \mapsto \quad u^\lambda(t,x) := \lambda^{-2}u(\lambda^{-3}t, \lambda^{-1}x), \qquad \lambda > 0.$$

If the datum satisfies $\|\mathcal{I}u(0)\|_{L^2} \le M$, then we first rescale it so that

$$\|\mathcal{I}u^\lambda(0)\|_{L^2} \le CM\lambda^{-\frac{3}{2}-s}N^{-s} = 1 \quad \Leftrightarrow \quad \lambda \sim (MN^{-s})^{\frac{2}{3+2s}},$$

and solve the equation from the rescaled datum. Rescaling back to the original one, we obtain a solution on the time interval $[0, O(\lambda^{-3}N^{\frac{3}{4}-\varepsilon})]$. Therefore, we can solve the equation up to an arbitrarily large time, by taking N sufficiently large, if $\lim_{N\to\infty} \lambda^{-3}N^{\frac{3}{4}-\varepsilon} = \infty$. This condition is equivalent to $\frac{-6s}{3+2s} < \frac{3}{4}$, or $s > -3/10$, and GWP for these s follows.

To show GWP in $H^{-\frac{3}{4}+\varepsilon}$, we have to add some correction terms to the almost conserved quantity $\|\mathcal{I}u(t)\|_{L^2}$ and improve the decay with respect to N in Lemma 2. See [7] for details.

1.3 Main results

As seen above, the LWP theory for the KdV and the modified KdV equations has been completed in some sense. However, we point out that the above result for KdV in $H^{-3/4}$ is relatively weak, compared with that for $s > -3/4$. Firstly, the uniqueness of solutions was obtained only in the image of the Miura transform. Recall that in the case $s > -3/4$, it was shown that solutions are unique in the Bourgain space $X^{s,b}$. We find it difficult to verify whether a given function is in the Miura image or not. Secondly, we do not have the control of their local solutions in a function space well adapted to the I-method, such as $X^{s,b}$. This is why GWP for real-valued (1) in $H^{-3/4}$ and GWP for (2) in $H^{1/4}$ were left open.

From this point of view, it is quite interesting to investigate the strong LWP for (1) in $H^{-3/4}$ by the iteration method. Our main result precisely deals with this issue. Of course, we have to change the working space from $X^{s,b}$. Our function space X will be some Besov-like generalization of the Bourgain space $X^{-3/4,1/2}$ with slight modification in low frequency (see the definition in the next section).

Theorem 1 ([19]). *The Cauchy problem* (1) *is locally well-posed in* $H^{-3/4}(\mathbb{R})$. *More precisely, for any* $r > 0$ *there exists a Lipschitz continuous data-to-solution map from a ball* $\{ u_0 \in H^{-3/4}(\mathbb{R}) \mid \|u_0\|_{H^{-3/4}} \le r \}$ *into* $X_T \cap C_t([0,T]; H_x^{-3/4}(\mathbb{R}))$ *for some* $T = T(r) > 0$ *(X_T is the time restricted space of X). Moreover, the solutions of* (1) *on the time interval* $[0,T]$ *are unique in the class* X_T.

To prove the uniqueness, we follow the argument given in [22] for quadratic nonlinear Schrödinger equations. This theorem, combined with the I-method (and the Miura transform), yields the global results for real-valued KdV (and mKdV). The proof is almost identical with the case of $X^{s,b}$ for $s > -3/4$, because our function space X is very close to the usual Bourgain spaces (in fact satisfies the embedding $X^{-3/4,b} \hookrightarrow X \hookrightarrow X^{-3/4,1/2}$ for any $b > 1/2$).

Corollary 1. *The Cauchy problem* (1) *is globally well-posed in* $H^{-3/4}(\mathbb{R})$ *in the real-valued case, and the Cauchy problem* (2) *(focusing or defocusing) is globally well-posed in* $H^{1/4}(\mathbb{R})$.

Note that in Lemma 1, the time of local existence is determined by $\int |\mathcal{I}u(t)|^2\,dx$, which is *not* agree with almost conserved quantity $\int (\mathcal{I}u(t))^2\,dx$ for the *complex-valued* KdV. Nonetheless, we can also have the global well-posedness for the *focusing* modified KdV, which is transformed into the complex-valued KdV by the Miura transform. For details, we refer to [7].

In the next section, we discuss how to construct the space X which yields the crucial bilinear estimate. For the entire proof of the desired estimates and the main theorem, see [19].

2 Construction of working space

As mentioned above, our main task to prove LWP is the construction of spacetime function space satisfying the linear and the bilinear estimates, since the Bourgain spaces $X^{-3/4,b}$ no longer yield the bilinear estimate. In this section, we shall recall some counterexamples to it, and then see how to modify the Bourgain spaces so that these examples may be suitably controlled.

We first prepare some notations for convenience. Let us fix a smooth function $q_0 : \mathbb{R} \to [0,1]$ which is equal to 1 on $[-5/4, 5/4]$ and supported in $[-8/5, 8/5]$. For $N > 0$ and $j = 1, 2, \ldots$, define

$$q_N(\xi) := q_0(\tfrac{\xi}{N}) - q_0(\tfrac{2\xi}{N}), \quad p_0 := q_0, \quad p_j := q_{2^j},$$

and then denote the Fourier multipliers with respect to x corresponding to q_0, q_N, p_0, and p_j by Q_0, Q_N, P_0, and P_j, respectively. Note that $\{ P_j \}_{j=0}^{\infty}$ is an inhomogeneous Littlewood-Paley decomposition, and that Q_N with $N > 0$ is the frequency-localizing operator satisfying supp $q_N \subset \{ \frac{5N}{8} \le |\xi| \le \frac{8N}{5} \}$, $q_N \equiv 1$ on $\{ \frac{4N}{5} \le |\xi| \le \frac{5N}{4} \}$.

2.1 Counterexamples to $X^{s,b}$- bilinear estimate; two nonlinear interactions

We begin with the following examples.

Proposition 1 ([16, 20])**.** *Let $b \in \mathbb{R}$. Then, there exists $c > 0$ such that the following holds.*

(i) For any $N \gg 1$, there exist $u_N, v_N \in \mathcal{S}(\mathbb{R}^2_{t,x})$ satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and

$$\big\|Q_0 \partial_x (u_N v_N)\big\|_{X^{-3/4,b-1}} \geq c N^{\frac{3}{2}b - \frac{3}{4}} \big\|u_N\big\|_{X^{-3/4,b}} \big\|v_N\big\|_{X^{-3/4,b}}.$$

(ii) For any $N \gg 1$, there exist $u_N, v_N \in \mathcal{S}(\mathbb{R}^2_{t,x})$ satisfying $Q_N u_N = u_N$, $Q_0 v_N = v_N$, and

$$\big\|\partial_x (u_N v_N)\big\|_{X^{-3/4,b-1}} \geq c N^{\frac{3}{4}} \langle N^{\frac{3}{2}b} \rangle^{-1} \big\|u_N\big\|_{X^{-3/4,b}} \big\|v_N\big\|_{X^{-3/4,b}}.$$

This proposition says that the bilinear estimate in $X^{-3/4,b}$

$$\big\|\partial_x (uv)\big\|_{X^{-3/4,b-1}} \leq C \big\|u\big\|_{X^{-3/4,b}} \big\|v\big\|_{X^{-3/4,b}} \tag{3}$$

fails to hold for $b > 1/2$ by (i), and for $b < 1/2$ by (ii).

We shall omit the precise definition of u_N and v_N, while they are sketched in Figure 1. We observe that the example in (i) consists of high-frequency functions supported in the frequency space along the curve $\tau = \xi^3$, and their product (or, in frequency, their convolution) is concentrated near the frequency origin (thus in the low-frequency region). We call such interactions *high-high-low*. On the other hand, the example in (ii) is the interaction between functions of high frequency and low frequency, which produces a high-frequency component near the curve, so we call it *high-low-high* interaction. We need to design the working space in which we can control these two typical interactions at a time.

The bilinear estimate (3) also fails in the case $b = 1/2$, but the divergence order is *logarithmic* in N rather than *power* in N as Proposition 1.

Proposition 2 ([23])**.** *Let $0 < \alpha < 1/2$. Then, there exists $c > 0$ such that the following holds: For any $N \gg 1$, there exist $u_N, v_N \in \mathcal{S}(\mathbb{R}^2_{t,x})$ satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and*

$$\big\|Q_0 \partial_x (u_N v_N)\big\|_{X^{-3/4,-1/2}} \geq c (\log N)^{\alpha} \big\|u_N\big\|_{X^{-3/4,1/2}} \big\|v_N\big\|_{X^{-3/4,1/2}}.$$

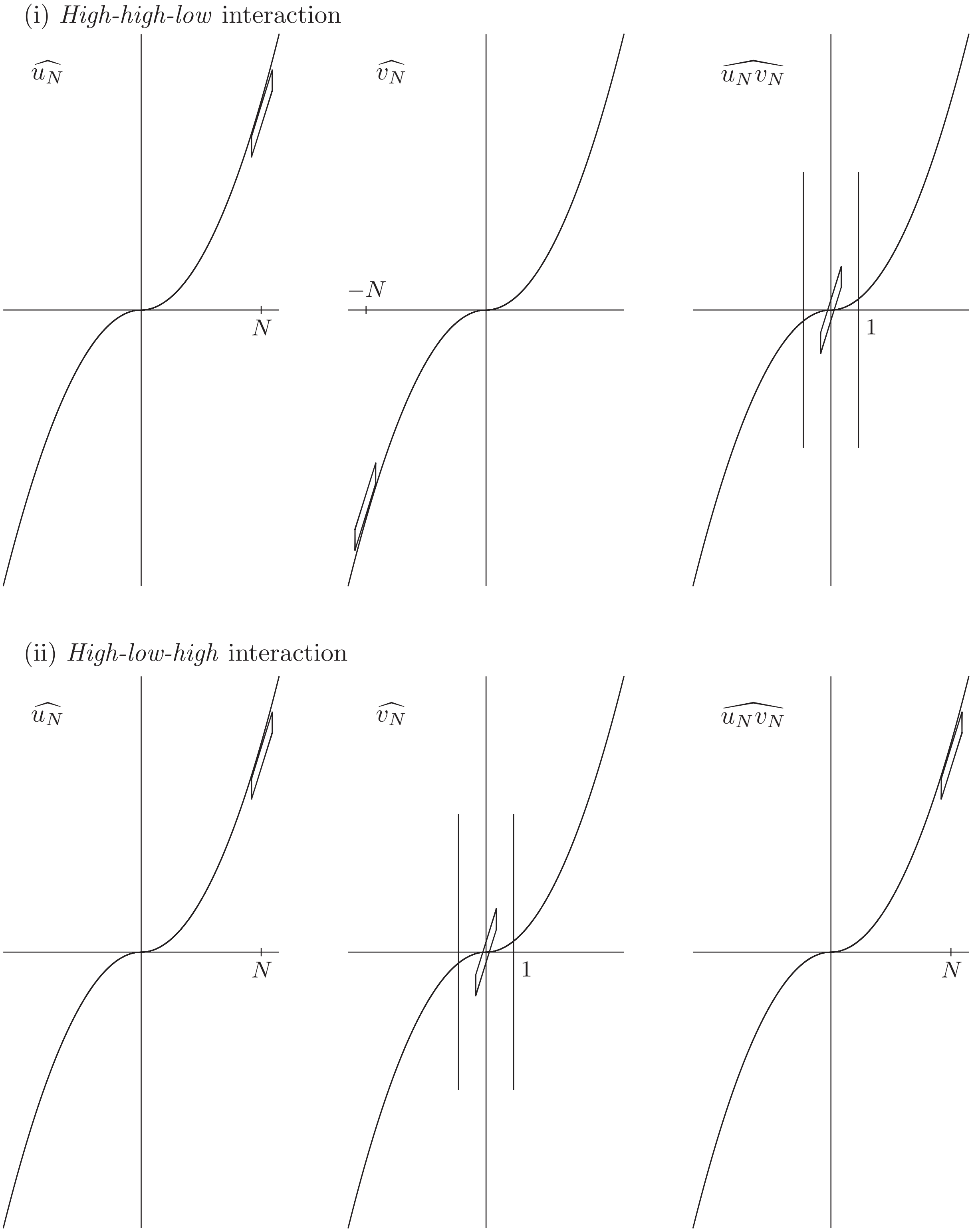


Figure 1. Two typical nonlinear interactions described in Proposition 1. In the context of the bilinear estimate (3) for $b \neq 1/2$, they produce some *power* divergences in N.

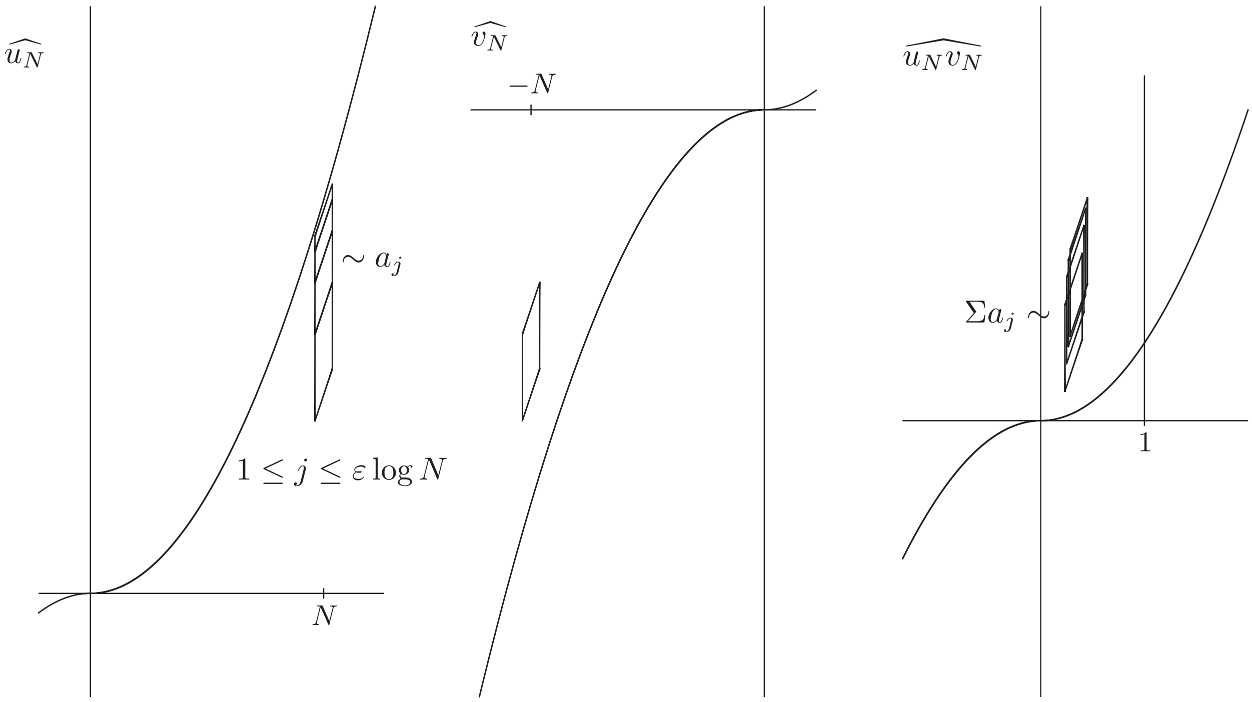


Figure 2. The example of high-high-low interaction described in Proposition 2, which breaks the bilinear estimate in $X^{-3/4,1/2}$ with *logarithmic* divergence.

As sketched in Figure 2, this example of high-high-low type is much more complicated than the previous one. We point out that the high-frequency function is supported also in the region distant from the curve $\tau = \xi^3$ in contrast to the counterexamples in Proposition 1. In fact, u_N consists of $\varepsilon \log N$ components dyadically supported away from the curve ($0 < \varepsilon \ll 1$). Each of these components $u_{N,j}$ ($1 \le j \le \varepsilon \log N$), which has some positive $X^{-3/4,1/2}$-norm a_j, interacts with v_N and outputs the component, whose norm is $\left\|\partial_x(u_{N,j}v_N)\right\|_{X^{-3/4,-1/2}} \gtrsim a_j \left\|v_N\right\|_{X^{-3/4,1/2}}$, at almost the *same* part of the low-frequency region $\{\, |\xi| \le 1 \,\}$. The norm of the total output is then at least $\left\|v_N\right\|_{X^{-3/4,1/2}} \Sigma a_j$, while the norm of u_N is equal to the ℓ^2-sum of those of $u_{N,j}$'s; $\left\|u_N\right\|_{X^{-3/4,1/2}} \sim (\Sigma a_j^2)^{1/2}$. Putting $a_j = j^{\alpha-1}$ ($0 < \alpha < 1/2$) for instance, we have the divergence of $O((\log N)^\alpha)$.

2.2 Modification in high frequency; Besov-type $X^{s,b}$ spaces

We have seen that the bilinear estimate in $X^{-3/4,b}$, (3), fails for any $b \in \mathbb{R}$, and that the divergence in the case $b = 1/2$ is logarithmic, better than the other cases. Therefore, we shall start from $X^{-3/4,1/2}$ and modify it to endure the nonlinear interaction described in Proposition 2.

In the analysis with the Bourgain spaces, in fact, such logarithmic divergences of estimates often occur in such a “critical” regularity. (This may be different from the critical regularity with respect to the scaling.) One standard way to avoid this is the ℓ^1-*Besov* modification (in the temporal direction) of the Bourgain spaces. This is similar to the space $B^{1/2}_{2,1}(\mathbb{R})$ as a modification of $H^{1/2}(\mathbb{R})$, which has many good properties such as the embedding into the space of bounded continuous functions. Such Besov-type Bourgain spaces were used first in the context of the wave map equations ([26]), and have appeared in a number of literature. For instance, see [5, 11] for the Kadomtsev-Petviashvili equations, [4, 13, 10] for the Benjamin-Ono and related equations, [22, 25, 1, 18] for the (quadratic) nonlinear Schrödinger equations, and [12] for the Schrödinger map.

The ℓ^1-Besov Bourgain spaces $X^{s,b,1}$ is defined by the norm

$$\|u\|_{X^{s,b,1}} := \Big(\sum_{j=0}^{\infty} 2^{2sj} \Big(\sum_{k=0}^{\infty} 2^{bk} \|p_j(\xi)p_k(\tau - \xi^3)\widehat{u}\|_{L^2_{\tau,\xi}} \Big)^2 \Big)^{1/2}.$$

The usual $X^{s,b}$-norm is equivalent to the above norm with the ℓ^1_k-sum replaced by ℓ^2_k. Note that $X^{s,b,1}$ is slightly stronger than $X^{s,b}$. In what follows, for convenience, we write X_+ and X_- to denote $X^{-3/4,1/2,1}$ and $X^{-3/4,1/2}$, respectively. (We use “+” or “−” sign in relation to the “stronger” or the “weaker” structure.)

The counterexample in Proposition 2 can be well controlled if we measure the high-frequency function u_N in X_+, because its X_+-norm is now the ℓ^1-sum of a_j’s. When we work at the bottom regularity, similar issues may arise, and the space $X^{s,1/2,1}$ is considered generally as a good substitute for $X^{s,1/2}$.

However, no one could show the bilinear estimate in X_+ for the KdV equation. This fact suggests that, for the KdV case, one can not restore the bilinear estimate in $X^{-3/4,b}$ by just making the ℓ^1-Besov modification. That would be the main reason why this problem of much interest had been left open since the bilinear estimate for $s > -3/4$ was established in [16]. The second result of us, Proposition 3 below, clarifies that this difficulty is essential and the bilinear estimate in X_+ actually fails.

To control the nonlinear interaction described in Proposition 2, we decide to use the stronger X_+ structure in high frequency. Then, it seems natural to consider two candidates for our working space first; one is X_+ and the other is X_0 defined by

$$\|u\|_{X_0} := \|P_0 u\|_{X_-} + \|(1-P_0)u\|_{X_+}.$$

	$X^{-\frac{3}{4},\frac{1}{2}+\varepsilon}$	$X^{-\frac{3}{4},\frac{1}{2}-\varepsilon}$	$X^{-\frac{3}{4},\frac{1}{2}}$	X_+	X_0
high-high-low	N^α Prop 1 (i)		$(\log N)^\alpha$ Prop 2	$(\log N)^\alpha$ Prop 3 (i)	
high-low-high		N^α Prop 1 (ii)	$(\log N)^\alpha$ Prop 3 (ii)		$(\log N)^\alpha$ Prop 3 (ii)

Table 1. Various divergences in the bilinear estimates for $s = -3/4$.

The only difference between them is their structures in low frequency. Unfortunately, it turns out that neither X_+ nor X_0 yields the bilinear estimate, so the Besov modification is actually not sufficient by itself.

Proposition 3 ([20]). *Let $0 < \alpha < 1/2$. Then, there exists $c > 0$ such that the following holds:*

(i) For any $N \gg 1$, there exist $u_N, v_N \in \mathcal{S}(\mathbb{R}^2_{t,x})$ satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and

$$\big\| Q_0 \langle \partial_t + \partial_x^3 \rangle^{-1} \partial_x (u_N v_N) \big\|_{X_+} \geq c (\log N)^\alpha \big\| u_N \big\|_{X_+} \big\| v_N \big\|_{X_+},$$

where $\langle \partial_t + \partial_x^3 \rangle^{-1}$ is the Fourier multiplier with the symbol $\langle \tau - \xi^3 \rangle^{-1}$.

(ii) For any $N \gg 1$, there exist $u_N, v_N \in \mathcal{S}(\mathbb{R}^2_{t,x})$ satisfying $Q_N u_N = u_N$, $Q_0 v_N = v_N$, and

$$\big\| \langle \partial_t + \partial_x^3 \rangle^{-1} \partial_x (u_N v_N) \big\|_{X_+} \geq c (\log N)^\alpha \big\| u_N \big\|_{X_+} \big\| v_N \big\|_{X_-},$$

$$\big\| \langle \partial_t + \partial_x^3 \rangle^{-1} \partial_x (u_N v_N) \big\|_{X_-} \geq c (\log N)^\alpha \big\| u_N \big\|_{X_-} \big\| v_N \big\|_{X_-}.$$

Note that the bilinear estimates in X_+ and X_0 are disproved by the examples given in (i) and (ii) (with the first estimate), respectively. Also, in addition to Proposition 2, the second estimate in (ii) re-proves the failure of the bilinear estimate in $X_- = X^{-3/4,1/2}$. More precisely, (i) shows that low-frequency structure of the space X_+ is too strong to control the high-high-low interactions. However, if we replace it with the weaker structure X_- in low frequency and employ X_0, then it is too weak to control the high-low-high interactions. Moreover, the high-low-high interactions also break the bilinear estimate in X_-. Thus, we can sum up the divergences in bilinear estimates for the regularity $s = -3/4$ as Table 1.

2.3 Modification in low frequency

Let us look into details of examples given in Proposition 3, which are sketched in Figure 3 and precisely defined in [20].

For (i), $\varepsilon \log N$ components of the function u_N are all supported *near* the curve $\tau = \xi^3$, unlike the example given in Proposition 2. Thus, the stronger X_+-norm of u_N is still given by the ℓ^2-sum. On the other hand, we see that the output components are supported in the low-frequency region between $N^{-1/2}$ and 1, and also dyadically separated with respect to $\tau - \xi^3$. Thus, the norm of the output amounts to the ℓ^1-sum if we employ the ℓ^1-Besov structure in low frequency. Remark that all the ℓ^p_k-norms are equivalent if a function is restricted near the curve $\tau = \xi^3$ since there is no summation over k in such a case, and that the modification from ℓ^2_k to ℓ^1_k has an effect only when a function is supported away from the curve.

For (ii), the low-frequency function v_N has $\varepsilon \log N$ components between 0 and $N^{-1/2}$, and its X_--norm is given by the ℓ^2-sum. We see that outputs of the interaction of these components with u_N fall onto almost the same frequency position near the curve. Therefore, the norm of the output is the ℓ^1-sum, no matter which structure we use in high frequency.

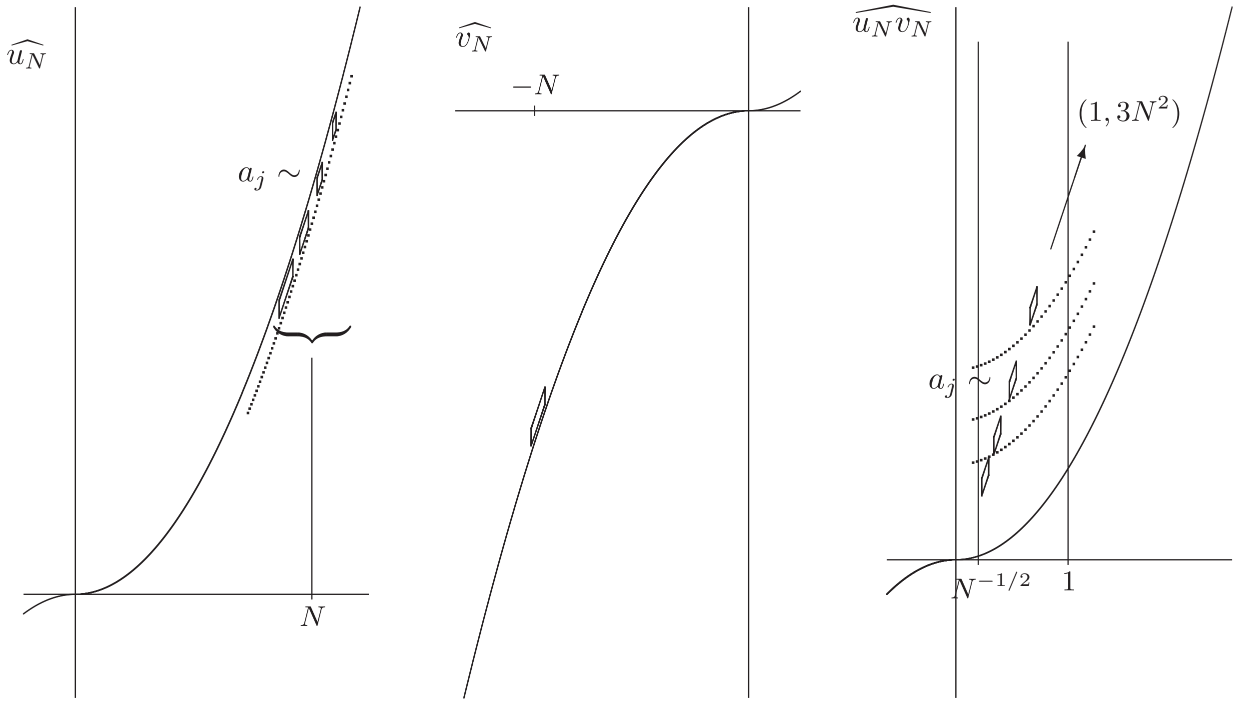
We thus have to find the norm weaker than X_+ and stronger than X_- in low frequency so that we can simultaneously control the high-high-low and the high-low-high interactions. It is then worth noting that two interactions in Proposition 3 come from different parts of the low-frequency region separated by the fuzzy boundary $|\xi| \sim N^{-1/2}$. This suggests that we may use X_+ in very low frequency $\{\, |\xi| \lesssim N^{-1/2} \,\}$ and use X_- in middle frequency $\{\, N^{-1/2} \lesssim |\xi| < 1 \,\}$. In fact, it turns out in the end that the high-high-low interaction can be controlled in very low frequency even if we assume the stronger structure X_+ there, and that the high-low-high can be still controlled under the weaker structure X_- in middle frequency.

Although the boundary $|\xi| \sim N^{-1/2}$ depends on N, we can exactly execute this idea by making use of another feature that both of two interactions in Proposition 3 come from the low-frequency region *along the line* $\tau = 3N^2\xi$. We first define $D \subset \mathbb{R}^2_{\tau,\xi}$ by

$$D := \{\, |\xi| < 1,\ |\tau| > |\xi|^{-3} \,\},$$

which is essentially equal to our 'middle-frequency' region $\{\, \tau \sim 3N^2\xi,\ N^{-1/2} \lesssim |\xi| < 1,\ N \gg 1 \,\}$ (see Figure 4). Then, the working space X is defined by

(i) Logarithmic divergence of the bilinear estimate in X_+ (*high-high-low*)



(ii) Logarithmic divergence of the bilinear estimate in X_0 or $X^{-3/4,1/2}$ (*high-low-high*)

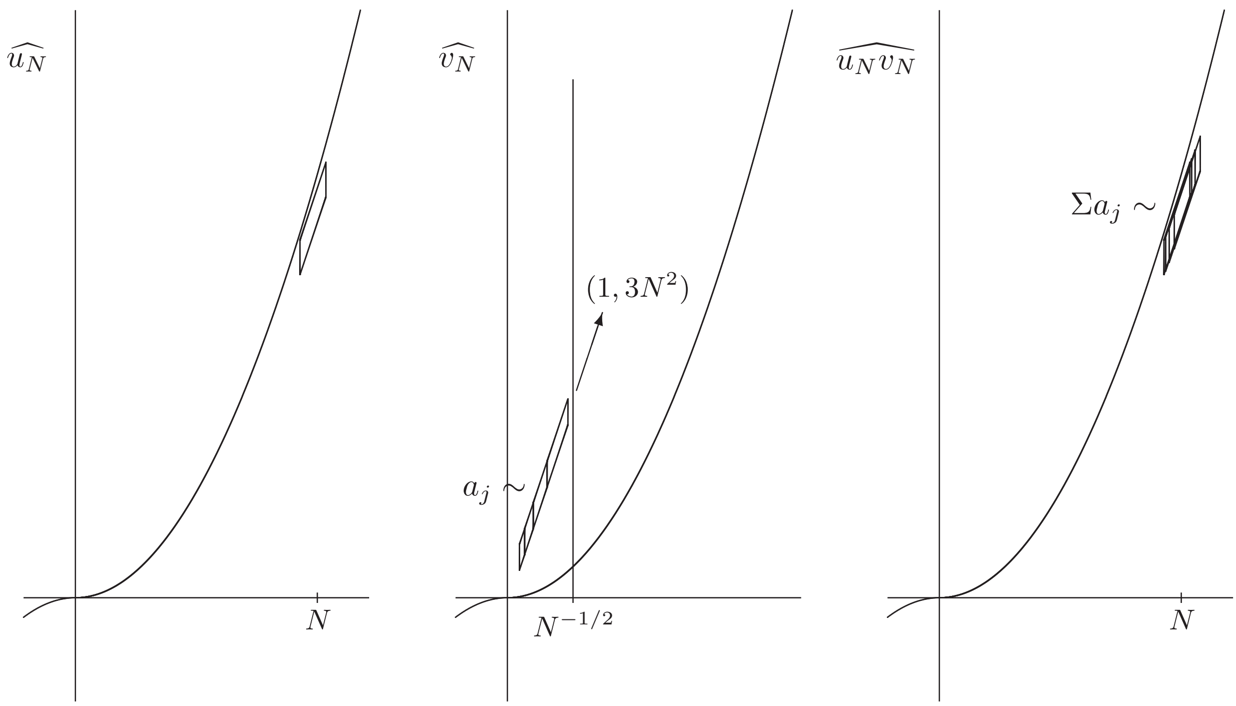


Figure 3. Counterexamples to the bilinear estimates described in Proposition 3.

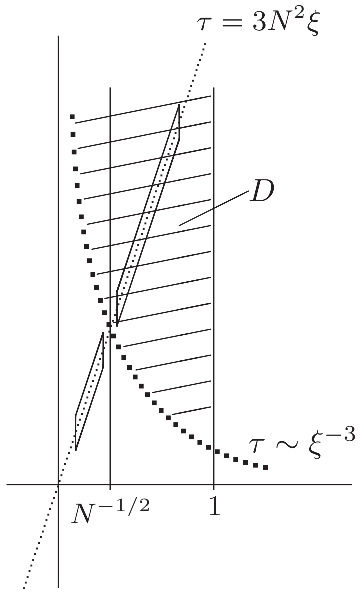


Figure 4. The set D is essentially equal to our 'middle-frequency' region.

$$\|u\|_X := \|P_D u\|_{X_-} + \|(1-P_D)u\|_{X_+},$$

where P_D is the frequency-localizing operator to the set D. (Since the interactions in Proposition 3 only come from the first and third quadrants, it may be more natural to use weaker norm in the region $D \cap \{\tau\xi > 0\}$. One can also show the bilinear estimate from this definition, because it does not make any difference which of X_+ and X_- we use in the second and fourth quadrants of the low frequency region. We take the above definition of the set D just for the sake of simplicity.)

For this X we can establish the desired bilinear estimate

$$\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(uv)\|_X \le C\|u\|_X\|v\|_X,$$

as well as linear estimates. However, X_- is not embedded into $C_t(H_x^{-3/4})$, and nor is X. Then, we introduce an auxiliary space Y defined by $\|u\|_Y := \|\langle\xi\rangle^{-3/4}\widehat{u}\|_{L^2_\xi(L^1_\tau)}$. The space Y has also appeared in previous works, originally in [8]. We can prove a similar bilinear estimate for Y,

$$\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(uv)\|_Y \le C\|u\|_X\|v\|_X.$$

Since $X_+ \hookrightarrow Y \hookrightarrow C_t(H_x^{-3/4})$, in proving the above estimate, we may restrict the left hand side to the middle-frequency region D where we use the weaker norm X_-. The proof is in fact much easier than that of the bilinear estimate in X. These estimates enable us to apply the iteration method in the space $X \cap C_t(H_x^{-3/4})$; see [19] for details.

2.4 Remarks

Recently, Guo [9] obtained the same well-posedness results independently. The iteration method was also used in a modification of the Bourgain space, which is identical with ours in high frequency. The only difference is in low frequency $\{\, |\xi| \leq 1\,\}$. The space in [9] has the maximal function norm $\|u\|_{L_x^2 L_t^\infty}$, while our space has the norm of

$$\|P_D u\|_{X^{0,1/2}} + \|(1-P_D)u\|_{X^{0,1/2,1}} \quad (\ + \|u\|_{L_t^\infty L_x^2}\ ).$$

These structures share some common properties. For instance, both are weaker than $X_+ = X^{-3/4,1/2,1}$, which is for the high frequency part, and stronger than $C_t(H_x^{-3/4})$. However, there is no inclusion relation between two spaces. One of the features of our space X is that it is defined totally on the frequency space $\mathbb{R}^2_{\tau,\xi}$ similarly to the standard Bourgain spaces, in contrast to the space in [9] defined on the physical space $\mathbb{R}^2_{t,x}$ in low frequency. Such structure is compatible with the I-method and admits the identical proof for the global well-posedness as the previous one ([7]) working on the standard $X^{s,b}$ with $s > -3/4$. See Remark 2.3 in [19] for further comments.

Acknowledgment

The author would like to express his great appreciation to Professor Yoshio Tsutsumi for encouraging him to address this issue and giving him a number of valuable advices. He also offers his thanks to Professor Hideo Takaoka for informing him of the work of Guo. Finally, the author shows his deep gratitude to the referee for reading the manuscript carefully and making a perfect list of corrections including many precious suggestions. This work was partially supported by Grant-in-Aid for JSPS Fellows 20-2196.

References

[1] I. Bejenaru and T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. **233** (2006), 228-259.

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II*, Geom. Funct. Anal. **3** (1993), 209–262.

[3] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. **125** (2003), 1235–1293.

[4] J. Colliander, C.E. Kenig, and G. Staffilani, *Local well-posedness for dispersion-generalized Benjamin-Ono equations*, Differential Integral Equations **16** (2003), 1441–1472.

[5] ______, *Low regularity solutions for the Kadomtsev-Petviashvili I equation*, Geom. Funct. Anal. **13** (2003), 737–794.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness for KdV in Sobolev spaces of negative index*, Electron. J. Differential Equations **2001**, No. 26, 1–7.

[7] ______, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), 705–749.

[8] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), 384–436.

[9] Z. Guo, *Global well-posedness of Korteweg-de Vries equation in* $H^{-3/4}(\mathbb{R})$, J. Math. Pures Appl. **91** (2009), 583–597.

[10] ______, *Local well-posedness and a priori bounds for the modified Benjamin-Ono equation without using a gauge transformation*, preprint. (arXiv:0807.3764)

[11] M. Hadac, S. Herr, and H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 917–941.

[12] A.D. Ionescu and C.E. Kenig, *Low-regularity Schrodinger maps*, Differential Integral Equations **19** (2006), 1271–1300.

[13] ______, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces*, J. Amer. Math. Soc. **20** (2007), 753–798.

[14] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[15] ______, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993), 1–21.

[16] ______, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[17] ______, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), 617–633.

[18] N. Kishimoto, *Low-regularity bilinear estimates for a quadratic nonlinear Schrödinger equation*, J. Differential Equations **247** (2009), 1397–1439.

[19] ______, *Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity*, Differential Integral Equations **22** (2009), 447–464.

[20] ______, *Counterexamples to bilinear estimates for the Korteweg-de Vries equation in the Besov-type Bourgain space*, to appear in Funkcial. Ekvac.

[21] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. **39** (1895), 422–443.

[22] T. Muramatu and S. Taoka, *The initial value problem for the 1-D semilinear Schrödinger equation in Besov spaces*, J. Math. Soc. Japan **56** (2004), 853–888.

[23] K. Nakanishi, H. Takaoka, and Y. Tsutsumi, *Counterexamples to bilinear estimates related with the KdV equation and the nonlinear Schrödinger equation*, Methods Appl. Anal. **8** (2001), 569–578.

[24] T. Tao, "Nonlinear Dispersive Equations: Local and Global Analysis," CBMS Regional Conference Series in Mathematics 106, the American Mathematical Society, Providence RI, 2006. (ISBN: 0-8218-4143-2)

[25] S. Taoka, *Well-posedness of the Cauchy problem for the semilinear Schrödinger equation with quadratic nonlinearity in Besov spaces*, Hokkaido Math. J. **34** (2005), 65–96.

[26] D. Tataru, *Local and global results for wave maps, I*, Comm. Partial Differential Equations **23** (1998), 1781–1793.

E-mail address : `n-kishi@math.kyoto-u.ac.jp`